

On the calculation of the eigenvalues of one-dimensional anharmonic oscillators

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Abstract

We draw attention on the fact that the Riccati-Padé method developed some time ago enables the accurate calculation of bound-state eigenvalues as well as of resonances embedded either in the continuum or in the discrete spectrum. We apply the approach to several one-dimensional models that exhibit different kind of spectra. In particular we test a WKB formula for the imaginary part of the resonance in the discrete spectrum of a three-well potential.

Key words: Anharmonic oscillators, bound-state energies, resonances, Riccati-Padé method, Accurate calculations, WKB formula

1 Introduction

In a recent paper Gaudreau et al [1] proposed a method for the calculation of the eigenvalues of the Schrödinger equation for one-dimensional anharmonic oscillators. In their analysis of some of the many approaches proposed earlier with that purpose they resorted to expressions of the form: “However, the

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existing numerical methods are mostly case specific and lack uniformity when faced with a general problem.” “As can be seen by the numerous approaches which have been developed to solve this problem, there is a beautiful diversity yet lack of uniformity in its resolution. While several of these methods yield excellent results for specific cases, it would be favorable to have one general method that could handle any anharmonic potential while being capable of computing efficiently approximations of eigenvalues to a high pre-determined accuracy.” “Various methods have been used to calculate the energy eigenvalues of quantum anharmonic oscillators given a specific set of parameters. While several of these methods yield excellent results for specific cases, there is a beautiful diversity yet lack of uniformity in the resolution of this problem.” The authors put forward an approach that they termed double exponential Sinc collocation method (DESCM) and reported results of remarkable accuracy for a wide variety of problems. In fact they stated that “In the present work, we use this method to compute energy eigenvalues of anharmonic oscillators to unprecedented accuracy” which may perhaps be true for some of the models chosen but not for other similar examples. For example, in an unpublished article Trott [2] obtained the ground-state energy of the anharmonic oscillator with potential $V(x) = x^4$ with more than 1000 accurate digits. His approach is based on the straightforward expansion of the wavefunction in a Taylor series about the origin.

One of the methods mentioned by Gaudreau et al [1] is the Riccati-Padé method (RPM) based on a rational approximation to the logarithmic derivative of the wavefunction that satisfies a well known Riccati equation [3,4]. In their brief analysis of the RPM the authors did not mention that this approach not only yields the bound-state eigenvalues but also the resonances embedded in the continuum [5]. What is more, the same RPM quantization condition, given by a Hankel determinant, produces the bound-state eigenvalues, the resonances embedded in the continuum as well as some kind of strange reso-

nances located in the discrete spectrum of some multiple-well oscillators [6]. It is not clear from the content of [1] whether the DESCМ is also suitable for the calculation of such complex eigenvalues.

The accuracy of the calculated eigenvalues not only depends on the chosen method but also on the available computation facilities and on the art of programming. For this reason the comparison of the accuracy of the results reported in a number of papers spread in time should be carried out with care.

The purpose of this paper is two-fold. First, we show that the RPM can in fact yield extremely accurate eigenvalues because it exhibits exponential convergence. To that end it is only necessary to program the quantization condition in an efficient way in a convenient platform. Second, we stress the fact that the RPM yields both real and complex eigenvalues with similar accuracy through the same quantization condition. More precisely: it is not necessary to modify the algorithm in order to obtain such apparently dissimilar types of eigenvalues that are associated to different boundary conditions of the eigensolution.

In section 2 we outline the RPM for even-parity potentials. In section 3 we apply this approach to some of the examples discussed by Gaudreau et al [1] and obtain eigenvalues with remarkable accuracy. In this section we also calculate several resonances supported by anharmonic oscillators that were not taken into account by those authors. We consider examples of resonances embedded in the continuous as well as in the discrete spectrum. Finally, in section 4 we summarize the main results and draw conclusions.

2 The Riccati-Padé method

The dimensionless Schrödinger equation for a one-dimensional model reads

$$\psi''(x) + [E - V(x)]\psi(x) = 0, \quad (1)$$

where E is the eigenvalue and $\psi(x)$ is the eigenfunction that satisfies some given boundary conditions. For example, $\lim_{|x| \rightarrow \infty} \psi(x) = 0$ determines the discrete spectrum and the resonances are associated to outgoing waves in each channel (for example, $\psi(x) \sim Ae^{ikx}$). In this paper we restrict ourselves to anharmonic oscillators with even-parity potentials $V(-x) = V(x)$ to facilitate the comparison with the results reported by Gaudreau et al [1] but it should be taken into account that the approach applies also to non-symmetric potentials [7].

In order to apply the RPM we define the regularized logarithmic derivative of the eigenfunction

$$f(x) = \frac{s}{x} - \frac{\psi'(x)}{\psi(x)}, \quad (2)$$

that satisfies the Riccati equation

$$f'(x) + \frac{2sf(x)}{x} - f(x)^2 + V(x) - E = 0, \quad (3)$$

where $s = 0$ or $s = 1$ for even or odd states, respectively. If $V(x)$ is a polynomial function of x or it can be expanded in a Taylor series about $x = 0$ then one can also expand $f(x)$ in a Taylor series about the origin

$$f(x) = x \sum_{j=0}^{\infty} f_j(E) x^{2j}. \quad (4)$$

On arguing as in earlier papers we conclude that we can obtain approximate eigenvalues to the Schrödinger equation from the roots of the Hankel deter-

minant

$$H_D^d(E) = \begin{vmatrix} f_{d+1} & f_{d+2} & \cdots & f_{d+D} \\ f_{d+2} & f_{d+3} & \cdots & f_{d+D+1} \\ \vdots & \vdots & \ddots & \vdots \\ f_{d+D} & f_{d+D+1} & \cdots & f_{d+D-1} \end{vmatrix} = 0, \quad (5)$$

where $D = 2, 3, \dots$ is the dimension of the determinant and d is the difference between the degrees of the polynomials in the numerator and denominator of the rational approximation to $f(x)$ [3–6]. In those earlier papers we have shown that there are sequences of roots $E^{[D,d]}$, $D = 2, 3, \dots$ of the determinant $H_D^d(E)$ that converge towards the bound states and resonances of the quantum-mechanical problem. We have at our disposal a set of sequences for each value of d but it is commonly sufficient to choose $d = 0$. For this reason, in this paper we restrict ourselves to the sequences of roots $E^{[D]} = E^{[D,0]}$ (unless stated otherwise).

In this paper we are concerned with anharmonic-oscillator potentials of the form

$$V(x) = \sum_{j=1}^K v_j x^{2j}. \quad (6)$$

The spectrum is discrete when $v_K > 0$ and continuous when $v_K < 0$. In the latter case there may be resonances embedded in the continuous spectrum which are complex eigenvalues. The real part of any such eigenvalue is the resonance position and the imaginary part is half its width Γ ($|\Im E| = \Gamma/2$).

3 Examples

Four examples chosen by Gaudreau et al [1] are quasi-exactly solvable problems; that is to say, one can obtain exact solutions for some states:

$$\begin{aligned}
V_1(x) &= x^2 - 4x^4 + x^6 & E_0 &= -2 \\
V_2(x) &= 4x^2 - 6x^4 + x^6 & E_1 &= -9 \\
V_3(x) &= \frac{105}{64}x^2 - \frac{43}{8}x^4 + x^6 - x^8 + x^{10} & E_0 &= \frac{3}{8} \\
V_4(x) &= \frac{169}{64}x^2 - \frac{59}{8}x^4 + x^6 - x^8 + x^{10} & E_1 &= \frac{9}{8}.
\end{aligned} \tag{7}$$

The RPM yields the exact result for all these particular cases because in all of them the logarithmic derivative $f(x)$ is a rational function of the coordinate.

The Hankel determinants of lowest dimension for each case are:

$$\begin{aligned}
H_2^0(E) &= \frac{1}{4725} (E + 2) (E^5 - 2EE^4 - 23EE^3 - 602EE^2 + 1030EE - 1412), \\
H_2^0(E) &= \frac{1}{4465125} (E + 9) (E^5 - 9E^4 - 187E^3 - 8217E^2 + 78336E - 348624), \\
H_3^0(E) &= \frac{1}{3189612751764848640000} (8E - 3) (8589934592E^{11} + 3221225472E^{10} \\
&\quad - 1887235473408E^9 - 399347250364416E^8 - 1634745666502656E^7 \\
&\quad + 10770225531715584E^6 - 836065166572191744E^5 \\
&\quad - 905684630058491904E^4 + 5197219286067104256E^3 - 2944302537136698432E^2 \\
&\quad - 12283878786837315912E + 22452709866105906693), \\
H_3^0(E) &= \frac{1}{431028319209742820966400000} (8E - 9) (8589934592E^{11} + 9663676416E^{10} \\
&\quad - 5569096187904E^9 - 2064531673055232E^8 - 15362232560910336E^7 \\
&\quad + 158709729905344512E^6 - 23752960275863896064E^5 \\
&\quad - 84068173973645402112E^4 + 2318080070178601634304E^3 \\
&\quad - 6274577633554290840768E^2 \\
&\quad - 75410626140297229262472E + 655367638076442656931879),
\end{aligned} \tag{8}$$

respectively. It is clear that the first factor of each Hankel determinant yields the exact eigenvalue of the corresponding model in equation (7).

As a nontrivial example we consider the quartic anharmonic oscillator

$$V(x) = x^2 + \lambda x^4. \quad (9)$$

Gaudreau et al [1] calculated the ground state for $\lambda = 1$ with remarkable accuracy. The RPM also enables great accuracy because of its exponential convergence. For example, with determinants of dimension $D \leq 623$ we obtained

$$\begin{aligned} E_0 = & 1.3923516415302918556575078766099341846000667112208340889063493 \\ & 238775674318756465285909735634677917591211513753417388174455516 \\ & 240463837130438178697370013460935168154842085748896569018003055 \\ & 412366487432189534357154174093826240572295199985687111814096892 \\ & 270227363816981111260310703429386134195964568485918291463489851 \\ & 885814863025469392145221031177208948219643654580541741801366088 \\ & 701870825264349698158700823407607595743192268511389600196854493 \\ & 949820962407561620946196334634473774557014921149262346890591637 \\ & 338563062681405570992510627058090950578666603093583144835197352 \\ & 905560061049224302849821825415119194035000689109989896675454979 \\ & 833183805654199754661625730310527294045815675292625382286721180 \\ & 760183199752945956111132457567844565301841956779850974931537225 \\ & 418858821696022599972698095084658065637021365447651793869049904 \\ & 755455309191949465274340562585980971938979595684138772300267900 \\ & 681776732778457086544772456313662681845199346441260519691501249 \\ & 723061727243936387451149975151714249881364996642295004595485151 \\ & 916507248813368615814421881730600039773536840117104637678735672 \\ & 726392478420532548924901523470626991951440934018875830719295468 \\ & 178231131253774713120042218812766794224608722685106067661795491 \\ & 30792640798558850522732484547554994100518213983 \end{aligned}$$

which is considerably more accurate than the result reported by those authors. Such an accuracy is unnecessary but it clearly shows the stability and remarkable rate of convergence of the RPM.

For some approaches the pure quartic oscillator

$$V(x) = x^4 \quad (10)$$

may be more demanding than the oscillator treated previously. However, this is not the case for the RPM that yields

$$E_0 = 1.060362090484182899647046016692663545515208728528977933216245 \\
24169594356304434442112689629913467170351054624435858252558087 \\
98082102931470131768363738249357892262460047081754469601416374 \\
88417282256905935757790888061788790263601549395690275196148900 \\
94293487358440944269489790121397146429095192335453382834703350 \\
57576151120257039888523720240221841103086573731091398915453658 \\
41031116794058335486000922744006963112670238862297142969961059 \\
215583226671376935508673610000831830027517926233573913906136180 \\
776498596961814994127928092728407079561060440722946809949136275 \\
729273872791368902798424722261716944488954751370438068405439187 \\
787729532342458743725431783231906038106874160440343745301468472 \\
781391861294047043103401351071607110353008929823275427661518986 \\
950565047160252756089526262191025688200964410287815640052705292 \\
932405076382650282591124773625384718547144025722854384852974504 \\
585709788402490669995704768445877091762029124375273254907116433 \\
440230294730692398190895685374535988446016002313291933059395869 \\
304916644281633946163324287004261461237743009952234204208597735 \\
690153565416850308941851348795734106585479719467596466796613467 \\
688586437952654519560568286715958338884743467012042420714918747 \\
871038429573389138985245894022263471696176996560440931170998547 \\
160646641857421281143088181114951122148431408871216620593130769 \\
234180229827246883626045356507913236221596486925870033200744409 \\
688064046239788178394698378070482686021742719460350750696191658 \\
224983009606134572666392863592176435340137189204481484648373028 \\
941252963863440446954353934473733433447707230478215508820964235 \\
1106900382833900237848230939194834$$

from determinants of dimension $D \leq 806$. We think that present result is more accurate than the one reported by Trott [2], the discrepancy being in the last 9 figures. In the case of the quartic anharmonic oscillators (6) and (9) it was proved that $E_0^{[D,0]}$ and $E_0^{[D,1]}$ are lower and upper bounds to the actual eigenvalue, respectively [3, 4]. In order to verify the accuracy of present calculation we verified that both bounds agreed to the last digit.

As stated in the introduction, the RPM yields not only the bound states but also the resonances. For example, in the case of the anharmonic oscillator (9) with $\lambda = -0.1$ (inverted double well) we obtain

$$\Re E = 0.900672904092015024804721689210287758304603316620306983171851692403487146394470249616726790089628688252937763774647325772577775029731774913514457447338587770926803971267801469755869917579845522510002054201377209951767663713270344807430710771398008790873985646991861751535214021599017867320106028797395394576658586997556297318927925824908161790321264175746331544068756411634377154244333590011870423051096533092590779249514766962853328430930211234477027796391708328562113084417269095730604423886606827779571527761692806042581009753079028908978267983213678426826574845962017573001105365337069958536171568094454228536129998896868782012308853651273647276896329380454199462228310270304944635915434440082426878311949185729315006099566578108882411457777224871083716564437160787964973794652063487913034212377042606399056515807797185750609934729619$$

and

$$|\Im E| = 0.006693280875800130269271875081318241122949894326169673589331409728261091560585043018393541639674672436214813574097441000697001863510171471544100944964712095259566361194259386325794519366933621549986695707277857784464014031443123369559867398037583541840546888210884788662488801718718713379746363683684612053685173456818977724162333280360677703301489912446298837896455581815166460445660555875185436390373393566787035417128350480863941836050449532217074896601384341982511525918760645238309783201470773284077821869851429564144374849598381665972161010695961186324314282177836539138159742843180320299235738874296270532861487213500896845311943281068341100885743370599489265739919032484937894055855027258242635681029615386184966870885537480837946534215566858502262251410323283978252816367927891368870401758500263860266656837355486939353652027140$$

with determinants of dimension $D \leq 429$. As far as we know this resonance was not calculated with such an accuracy before.

A most interesting example is given by

$$V(x) = x^2 \left(1 - g^{2k} x^{2k}\right)^2, \quad k = 1, 2, \dots \quad (11)$$

It is a three-well potential with minima $V(x_m) = 0$ at $x_m = -1/g, 0, 1/g$. Since $V(x)$ behaves asymptotically as $g^{4k} x^{4k+2}$ when $|x| \rightarrow \infty$ then one expects only bound states with positive eigenvalues when g is real. However, Benassi et al [8] proved that this family of potentials supports complex eigenvalues with all the properties of actual resonances. They calculated the lowest resonance for $k = 1$ and several values of g and compared the imaginary part with the WKB transmission coefficient through the barrier $|\Im E_{WKB}| \sim Ag^{-2}e^{-1/(2g^2)}$. Later Killingbeck [9] and Fernández [6] calculated this resonance more accurately and for smaller values of g and showed that $|\Im E| g^2 e^{1/(2g^2)}$ does not exhibit a uniform behaviour as suggested by the earlier calculation of Benassi et al [8]. In particular, Killingbeck [9] suggested that $|\Im E| g^2 e^{1/(2g^2)}$ exhibits a maximum. In this paper we calculated $\Im E$ even more accurately and for smaller values of g and present results, shown in Fig. 1, suggest that the conjecture that “the imaginary part of the resonance behaves as the WKB transmission coefficient through the barrier” may not be correct. The results of Benassi et al [8] for a shorter interval of g (also shown in the figure) give the impression that $|\Im E|$ has already reached the asymptotic behaviour $Ag^{-2}e^{-1/(2g^2)}$ which is not the case. The figure also shows the earlier results of Killingbeck [9] and Fernández [6].

The lowest bound state E_{bs} and the real part of the lowest resonance $\Re E_{res}$ approach each other as $g \rightarrow 0$ in such a way that $|E_{bs} - \Re E_{res}|$ is of the order of $|\Im E_{res}|$. This fact is clearly shown in Fig. 2.

The RPM enables us to calculate the bound states and resonances quite accurately. In what follows we show some of them for the potential (11) with $k = 1$ and $g = 0.2$.

By means of the RPM for even solutions and from determinants of dimension $D \leq 775$ we have estimated

$$E_{bs} = 0.932476291964221250713283307051702588320858910 \\
16450940452195530440541397037312615064902568796 \\
08959269025187554700854590142048116044798515032 \\
74607576315694596343111670051681020512461078632 \\
25947411410076408768061413506747150659467931140 \\
45773553564796578081916410057605227149530113848 \\
32488596806119963850647806580001262138356174439 \\
92627876704264450003234826765601226526080854117 \\
18883901908126394568677114253841158525832058185 \\
89486604967394261040369468954266459806798388982 \\
94552842394543958516859760723678410546834167750 \\
46265615981244952016961806157227974852224223062 \\
63307833915772881280212070865385935277969972309 \\
73792177447292258928727153726473008063005886824 \\
27244516489544916400797944748255960907744830950 \\
81889615946058064623691858375390646083072560381 \\
93215181635793814371703342844510993772509125579 \\
56334986610268220762863856194297627513825342515 \\
50679107510710529342767883873476401983945574715 \\
11757502147307390721534786949037416819236026830 \\
61521556243352973248410809257466846919735250978 \\
076767512840642337653806458904012731861910502$$

Analogously for the lowest resonance and from determinants of dimension $D \leq 691$ we obtained:

$$\Re E_{res} = 0.93255571582477452179676759062168990966452649573413 \\
85221674869167644974697878297482889666009195213742 \\
93717600069469906629572410914840067688495198625372 \\
17985690364409228164242506169354102397596504534361 \\
60103546722503111068374776347405701051766932138082 \\
85413600743099773501734379824559724078809846013142 \\
54466796141742916190093031117808393336341712037560 \\
74384668770561453730731352495106674855706489027647 \\
57628951678813837141930823450477006238516389060959 \\
38361501163923123468457211692027452357327627884344$$

99112499793762153016569474982638402568720610449675
 76070472867118837391778386061630035039234021063388
 55589439873306969552123357849387411686488958371712
 88731653433415162889450633825069899826979320533720
 96508988556846872196309982663487618177697150533924
 72202304048467020861359787434424955986376342243023
 12693029687970276633152983867273666099801795397355
 82538737743202827605750205148961320017464691543602
 10909829129113885230560751170580075050547310634884
 89167352701182508545578769602119439587479454959070
 96406141902643986905961993843552992164570091818648
 46232

and

$|\Im E_{res}| = 0.0000794775543996767650576037789218578987811077421$
 147849846562298140940287872947528146204563615614090
 028774445460294456021141163045800235709638364949663
 353905939799131542366777545481790495539496899144485
 754062579429110541064567909497805641816263780566675
 388233499495422056087561163648183005687447211370677
 159865123112232223161645886340846246513728783929916
 686176186086127723970824686813236167800836809517307
 934131065936050216138586041138997060636301660413252
 566742473343264345284070048369409457383472779415364
 601962826737820988546159296790667149898273960541718
 035505319649723627339124594304665614834862888093604
 238202929173352801815954037592272451961628205950969
 404764180454049311285199354566868017448385193114605
 883829806930057675979698384480925313088391011486261
 124472043838440003626344405429021299123105595053241
 661288059523168203490179201354209577049949175017998
 930282112536345053702570708768896012135190244497127
 005841328004652182923323394262461014032693652500467
 599419583388579441325365273718235875966252696289277
 9223128751255675814299886872626419368

There are also resonances among the odd solutions that, as far as we know, have not been reported so far. For example, from determinants of dimension

$D \leq 675$ we have estimated

$\Re E = 2.61567434444732550869411505135245734080470814119$
8081336401725389930302231715543431391542639715932
6112732616783541689552149339774924576459116223527
4031645919333724417808271441248941817228663854017
1598561805645280654160019475399671189366467557187
7023231933795457998655216288018422680281312142150
0900102213105491896812637673117118610977610908627
3339313659663532602566940509690710619685623673577
4702051685651830157091563631235768241343388226233
4214966804682833763697032153747063887197465955131
0578860140998664952511675889072946426592233312325
6683696960592694142085005517688414623839393719684
4660149637044379455905460056628569763686110064049
1982394293180159214983414747793630741582080077825
7407910054254871310091689731748516412820369579467
3252836879151952555347108138232897278070434422830
0626704068364463061804095353910257042206327201629
1297397160490735553290365416147019439703881706172
7015605838456405956320547199197731222371367961928
2365097186125212629398582930268581704082229677155
6014934576275263959253938748903578577589148738386
671

and

$|\Im E| = 0.012103006054949689419532092547676879902909061831561$
14606655397483574310015080238580046511006197881958452
80231225742919968027487497671478032524753374025385051
78880120535081995629925326152262013101829054124399674
83243664748674923025125583004541265899351700981055825
50256507625482608264912298496868223692732502872218110
91176377769035782652477078300369066474079388696220057
74215632091529416042979012944093555392893071799032327
21645370003682922872620954763793543027651537371155188
04371553808569321951667977345154403280617508551577861
29445926375665988730330512036322261513744864521271324
70480449317847540296878526243141260350713152579083265
00685413450993979606171778329687752598902925556571014

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11910087383972814671824614174682165252936046723308704
7280261989144813328509313

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The rate of convergence of the RPM for the bound states and the resonances may be different. One way of monitoring the rate of convergence is to calculate $\log |E^{[D]} - E^{[D-1]}|$ which is a straight line when the rate of convergence is exponential. In the present case we fit $y(D) = a + bD$ to $\log |E^{[D]} - E^{[D-1]}|$ and obtain $b(g)$ for several values of g . The results for the lowest even bound state and resonance are shown in figure 3. That figure clearly shows that the rate of convergence for the resonance is almost constant whereas it decreases with g for the bound state. As a result it is possible to obtain the resonance for small values of g , say $g < 1$, more accurately by means of determinants of similar and even lesser dimension. However, this advantage is counterbalanced by the fact that the mathematical operations require more CPU time when complex numbers are involved.

In order to carry out the necessary arithmetic of complex numbers with arbitrarily high precision we resorted to the GNU MPC C library [10] and to the recurrence relation:

$$H_D^d = \frac{H_{D-1}^d H_{D-1}^{d+2} - (H_{D-1}^{d+1})^2}{H_{D-2}^{d+2}}, \quad (12)$$

for a fast calculation of the Hankel determinants.

4 Conclusions

Throughout this paper we tried to stress two points. The first one is that the RPM can yield eigenvalues of remarkable accuracy if the algorithm is programmed judiciously. To this end we have calculated the lowest eigenvalues of the oscillators (9) (with $\lambda = 1$) and (10) with greater accuracy than those reported by Gaudreau et al [1] and Trott [2], respectively.

The second point is that only one RPM quantization condition applies to bound states and resonances. To illustrate it we calculated the resonances for two models with great accuracy. One of them is an ordinary resonance embedded in the continuum and other one is some kind of strange resonance appearing in the point spectrum close to the ground state. Present results for the lowest resonance in the discrete spectrum of the three-well potential (11) give support to the conjecture that the analytical WKB formula for the resonance width derived several years ago [8] may not be correct. Present results improve considerably upon those reported earlier by Killingbeck [9] and Fernández [2]. It is not clear to us whether the DESCМ [1] or the power series approach [2] may also be applied to resonances without considerable modification of the calculation algorithms.

It is not our purpose to criticise the DESCМ which is clearly a powerful algorithm as already proved by the remarkable calculations carried out by Gaudreau et al [1] on a wide variety of one-dimensional models. We just wanted to draw attention to some advantages of the RPM that have been overlooked in the discussion of the method carried out by those authors.

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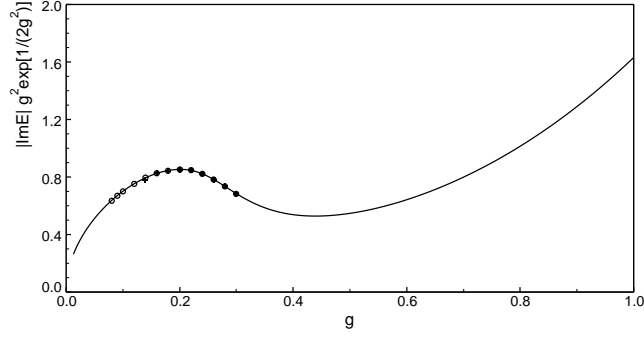


Fig. 1. Present calculation of $|\Im E| g^2 \exp[1/(2g^2)]$ for the oscillator (11) (solid line) and the results of Beanassi et al [8] (filled circles), Killingbeck [9] (crosses) and Fernández [6] (empty circles).

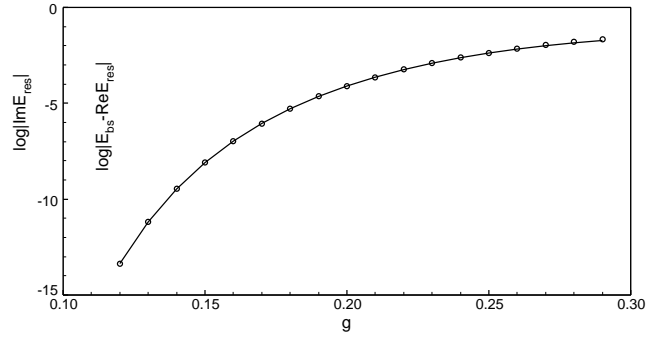


Fig. 2. $\log |\Im E_{res}|$ (circles) and $\log |\Re E_{res} - E_{bs}|$ (solid line) vs. g for the oscillator (11) with $k = 1$.

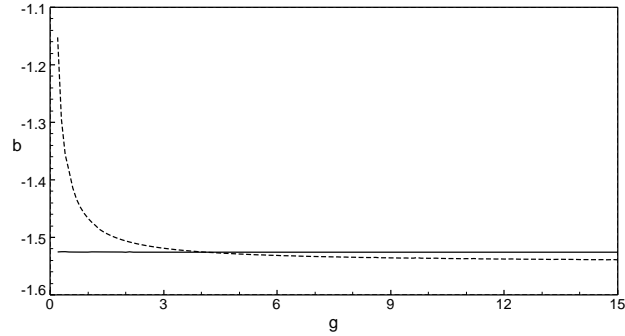


Fig. 3. Slope $b(g)$ for the lowest even bound state (dashed line) and resonance (solid line) of the oscillator (11).